CONSTRUCTION OF SHELL IN NONSYMMETRIC GRAVITY THEORY

Dobrowolski T. and Koc P.

Varna 2014

Plan of the lecture

- Introduction
  - Singularities in theory of GR
  - Mathematical motivation
  - Physical motivation
  - History of the NGT
  - Stability of physical vacuum
- The field equations in NGT
- The shell in NGT
  - Spherical symmetry
  - Strategy of constructing of the shell

  - Infinitesimally small shell

  - Energy - momentum tensor of the shell

- Summary

Introduction

Main problem of Einstein Theory of Gravity: the existence of singularities.

How to overcome this problem?

1) The evolution of the system is well defined outside of the event horizons. The validity of this approach needs the proof of the Cosmic Censorship Conjecture.

2)All singularities disappear at the quantum level of the gravity theory.

3) The last approach is based on the conviction that there is possibility to construct classical gravity theory that is free of singularities.

One of proposals of such a theory is based on non-Riemannian geometry, in which the metric tensor $g_{\mu\nu}$ can be split into symmetric and skew-symmetric part.

◇ Eisenhart L.P., *Non-Riemanian Geometry*, American Mathematical Society, New York, 1927.

Mathematical motivation

From mathematical point of view non-Riemannian geometry enables to circumvent assumptions of the Hawking and Penrose singularity theorems

(i) Einstein's equations holds.

(ii) The energy condition is satisfied.

(iii) There are no closed timelike curves.

$\diamond$ Hawking S.W. and Penrose R., Proc. Roy. Soc. London A **314** (1970) 529.

Physical motivation

If the metric tensor posses symmetric $g_{(\mu\nu)}$ and skew-symmetric part $g_{[\mu\nu]}$ as well

$$g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]}, \tag{1}$$

then the distances are not affected by the antisymmetric part of the metric tensor i.e.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{(\mu\nu)} dx^\mu dx^\nu. \tag{2}$$

On the other hand, the skew-symmetric part affects the determinant i.e. volume element of the space.

In these conditions there is a hope that matter in the sphere of fixed radius (identical with the radius of the sphere in General Relativity Theory) can form enough volume to prevent matter from formation of the black hole.

History of the NGT

First nonsymmetric theory was formulated by Einstein and then developed by Schrödinger and Straus. Einstein was motivated by desire of constructing a Unified Field Theory of classical gravity and electrodynamics.

◇ Einstein A., Ann. Math. **46** (1945) 578;
◇ A.Einstein, E.G.Straus, Ann. Math. **47** (1946) 731;
◇ Einstein A., Rev. Mod. Phys. **20** (1948) 35;
◇ Schrödinger E., *Space-Time Structure*, Cambridge University Press, Cambridge, 1954;
◇ Einstein A., *The Meaning of Relativity*, 5th ed., Princeton University Press, 1955.

The interest in nonsymmetric gravity has been restored by Moffat:

◇ Moffat J.W., Phys. Rev. D **19** (1979) 3554–3558;
◇ Moffat J.W., J. Math. Phys. **21** (1980) 1798.

$$\mathcal{L} = \sqrt{-g}\left(g^{\alpha\beta}R_{\alpha\beta} - 2\lambda\right) \qquad (3)$$

In 1993 Damour et al showed that wide class of Nonsymmetric Gravity Theories (NGT) posses serious consistency problems. They proved that a generic nonsymmetric model is not free of negative energy excitations (”ghosts”) and posses some algebraic inconsistencies.

◇ Damour T., Deser S., McCarthy J., Phys. Rev. D **47** (1993) 1541.

Finally, consistent gravity theory with nonsymmetric metric tensor has been formulated in paper:

◇ Moffat J.W., preprint **UTPT-94-30**.
◇ Moffat J.W., J. Math. Phys. **36** (1995) 3722.

$$\mathcal{L} = \sqrt{-g}\Big(g^{\alpha\beta}R_{\alpha\beta} - 2\lambda - \frac{1}{4}\mu^2 g^{\alpha\beta}g_{[\alpha\beta]} + \frac{1}{2}\sigma g^{\alpha\beta}W_\alpha W_\beta\Big) \qquad (4)$$

where: $W_\alpha = W^\beta_{\alpha\beta} - W^\beta_{\beta\alpha}$,

and $W^\gamma_{\alpha\beta}$ is nonsymmetric connection.

Stability of physical vacuum

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + \dots \tag{5}$$

$$\psi_{\mu\nu} \equiv h^{(1)}_{[\mu\nu]} \tag{6}$$

In lowest order the symmetric and skew-symmetric equations decouple. The symmetric equations are the Einstein field equations in the linear approximation. In addition to the symmetric equations (in the wave zone $T_{\mu\nu} = 0$) one can obtain the following field equations:

$$\left(\Box + \mu^2\right)\psi_{\alpha\beta} = 0, \quad \psi_\alpha = 0, \tag{7}$$

where $\psi_\alpha = \psi_{\alpha\beta,\gamma}\eta^{\gamma\beta}$.

On the other hand, the linearized field equations (7) follow from the lagrangian:

$$\mathcal{L}_{\psi} = \frac{1}{4}\psi_{\alpha\beta,\gamma}\psi^{\alpha\beta,\gamma} - \frac{1}{2}\psi_{\alpha}\psi^{\alpha} - \frac{1}{4}\mu^2\psi_{\alpha\beta}\psi^{\alpha\beta} \quad (8)$$

The equations (7) and therefore the lagrangian (8) were obtained under assumption that $\sigma = \frac{1}{3}$.

In flat Minkowski spacetime there exist only two physically acceptable models for the antisymmetric field $\psi_{\alpha\beta}$, which are free of tachyons, ghosts etc.

◇ van Nieuwenhuizen P., Nucl. Phys. B **60** (1973) 478.

One of these models is massive Proca-type model defined by the lagrangian (8). The theory defined by the lagrangian (8) is free of tachyons and its Hamiltonian is positive and bounded from below.

This means that in the NGT defined by the lagrangian (4) the physical vacuum is stable.

The field equations

$$G_{\mu\nu}(W)+\lambda g_{\mu\nu}+\frac{1}{4}\mu^2 C_{\mu\nu}+\frac{1}{2}\sigma(P_{\mu\nu}-\frac{1}{2}g_{\mu\nu}P) = 8\pi T_{\mu\nu}, \tag{9}$$

$$\mathbf{g}^{[\mu\nu]}{}_{,\nu} = 3\mathbf{D}^{\mu}, \tag{10}$$

$$\mathbf{g}^{\mu\nu}{}_{,\sigma}+\mathbf{g}^{\rho\nu}W^{\mu}_{\rho\sigma}+\mathbf{g}^{\mu\rho}W^{\nu}_{\sigma\rho}-\mathbf{g}^{\mu\nu}W^{\rho}_{\sigma\rho}+\frac{2}{3}\delta^{\nu}_{\sigma}\mathbf{g}^{\mu\rho}W^{\beta}_{[\rho\beta]}+$$

$$+\mathbf{D}^{\nu}\delta^{\mu}_{\sigma} - \mathbf{D}^{\mu}\delta^{\nu}_{\sigma} = 0, \tag{11}$$

where $W^{\mu}_{\rho\sigma}$ denotes the nonsymmetric connection. Additionally, field equations contain the Einstein like tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{12}$$

The other quantities present in equations of motion are defined below:

$$R_{\mu\nu}(W) = W^{\beta}_{\mu\nu,\beta} - \frac{1}{2}(W^{\beta}_{\mu\beta,\nu} + W^{\beta}_{\nu\beta,\mu})+$$

$$+W^{\beta}_{\alpha\beta}W^{\alpha}_{\mu\nu} - W^{\beta}_{\alpha\nu}W^{\alpha}_{\mu\beta}, \tag{13}$$

$$g^{\mu\nu} g_{\sigma\nu} = g^{\nu\mu} g_{\nu\sigma} = \delta^{\mu}_{\sigma}, \tag{14}$$

$$C_{\mu\nu} = g_{[\mu\nu]} + \frac{1}{2} g_{\mu\nu} g^{[\sigma\rho]} g_{[\rho\sigma]} + g^{[\sigma\rho]} g_{\mu\sigma} g_{\rho\nu}, \tag{15}$$

$$P_{\mu\nu} = W_\mu W_\nu, \qquad P = g^{\mu\nu} P_{\mu\nu}, \tag{16}$$

$$\mathbf{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}, \qquad \mathbf{D}^{\mu} = \frac{1}{2} \sigma \mathbf{g}^{(\mu\alpha)} W_\alpha. \tag{17}$$

Some simplification of the equation (9) can be achieved by introducing a new connection $\Gamma^{\lambda}_{\mu\nu}$:

$$\Gamma^{\lambda}_{\mu\nu} = W^{\lambda}_{\mu\nu} + \frac{2}{3}\delta^{\lambda}_{\mu}W_{\nu}. \tag{18}$$

New form of the equation (9) is the following:

$$\mathbf{g}^{\mu\nu}{}_{,\sigma} + \mathbf{g}^{\rho\nu}\Gamma^{\mu}_{\rho\sigma} + \mathbf{g}^{\mu\rho}\Gamma^{\nu}_{\sigma\rho} - \mathbf{g}^{\mu\nu}\Gamma^{\rho}_{(\sigma\rho)} + \mathbf{D}^{\nu}\delta^{\mu}_{\sigma} - \mathbf{D}^{\mu}\delta^{\nu}_{\sigma} = 0. \tag{19}$$

The contracted curvature tensor of the NGT can be related with Ricci-like tensor

$$R_{\mu\nu}(W) = R_{\mu\nu}(\Gamma) + \frac{2}{3}W_{[\mu,\nu]}. \tag{20}$$

Spherical symmetry

Significant simplification of the field equations is achieved in the case of spherical symmetry. The most general form of the metric tensor $g_{\mu\nu}$ in NGT has been derived in article

◇ Papapetrou A., Proc. R. Ir. Acad., Sec A **52** (1948) 69-86.

$$g_{\mu\nu} = \begin{pmatrix} \gamma & w & 0 & 0 \\ -w & -\alpha & 0 & 0 \\ 0 & 0 & -\beta & f\sin\theta \\ 0 & 0 & -f\sin\theta & -\beta\sin^2\theta \end{pmatrix}. \qquad (21)$$

Moreover, the assumption of the absence of the magnetic monopole implies that $w = 0$. The skew symmetric part of the metric in this case is therefore described by a single function f. In this case the fields W_μ, $P_{\mu\nu}$ and $\mathbf{D}^\mu$ become equal to zero:

$$W^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu}, \qquad P_{\mu\nu} = 0, \qquad \mathbf{D}^\mu = 0. \quad (22)$$

If we additionally put cosmological constants $\lambda = \mu = 0$ then the system of the field equations acquire a more familiar form

$$G_{\mu\nu}(\Gamma) = 8\pi T_{\mu\nu}, \qquad \mathbf{g}^{[\mu\nu]}{}_{,\nu} = 0, \tag{23}$$

$$g_{\mu\nu,\sigma} - g_{\rho\nu}\Gamma^{\rho}_{\mu\sigma} - g_{\mu\rho}\Gamma^{\rho}_{\sigma\nu} = 0. \tag{24}$$

Finally, we focus our interest on static field configuration and therefore the only non vanishing components of $G_{\mu\nu}$ have the form:

$$G_0{}^0 = -\frac{1}{2\alpha}A'' - \frac{1}{16\alpha}[3(A')^2 + 4B^2] + \frac{A'\alpha'}{4\alpha^2} + \frac{r^2}{r^4 + f^2}, \tag{25}$$

$$G_1{}^1 = \frac{1}{16\alpha}[-(A')^2 + 4B^2] - \frac{A'\gamma'}{4\alpha\gamma} + \frac{r^2}{r^4 + f^2}, \tag{26}$$

$$G_2{}^2 = G_3{}^3 = -\frac{\gamma''}{2\alpha\gamma} + \frac{\gamma'}{4\alpha\gamma}\left(\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma}\right) +$$

$$+\frac{A'}{8\alpha}\left(\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma}\right) - \frac{1}{4\alpha}A'' - \frac{1}{16\alpha}[(A')^2 + 4B^2], \tag{27}$$

$$G_3{}^2 = -sin^2\theta G_2{}^3 =$$

$$= sin\theta\left[\frac{f}{r^4 + f^2} + \frac{B}{4\alpha}\left(\frac{\gamma'}{\gamma} - \frac{\alpha'}{\alpha} + A'\right) + \frac{B'}{2\alpha}\right], \tag{28}$$

where

$$A = ln(r^4 + f^2), \qquad B = \frac{2fr - r^2 f'}{r^4 + f^2}, \tag{29}$$

and we made the standard choice of $\beta = r^2$.

Spherically symmetric shell The only known asymptotically flat solution with spherical symmetry of the Nonsymmetric Gravity Theory was found by Wyman.

◇ Wyman M., Can. J. Math. **2** (1950) 427.

In terms of the metric components (21), this solution can be written in the following way:

$$\gamma = e^{\nu}, \tag{30}$$

$$\alpha = \frac{e^{-\nu} m^2 (\nu')^2 (1 + s^2)}{[\cosh(a\nu) - \cos(b\nu)]^2}, \tag{31}$$

$$f = \frac{2m^2 e^{-\nu}[\sinh(a\nu)\sin(b\nu) + s(1 - \cosh(a\nu)\cos(b\nu))]}{[\cosh(a\nu) - \cos(b\nu)]^2}, \tag{32}$$

where

$$a^2 = \frac{\sqrt{1+s^2}+1}{2}, \qquad b^2 = \frac{\sqrt{1+s^2}-1}{2}. \tag{33}$$

Prime denotes differentiation with respect to space variable r. The function $\nu = \nu(r,s)$ is implicitly determined by the equation:

$$e^{\nu}[\cosh(a\nu) - \cos(b\nu)]^2 \frac{r^2}{2m^2} =$$
$$= \cosh(a\nu)\cos(b\nu) + s\sinh(a\nu)\sin(b\nu) - 1, \tag{34}$$

where s is dimensionless constant of integration.

For small r (i.e. for $r/2m \ll 1$) and for $|s| < 1$ the solution can be expanded in radial variable. The leading terms of this expansion are the following:

$$\gamma = \xi + \frac{\xi}{2|s|}\left(\frac{r}{m}\right)^2, \qquad \alpha = \frac{4\xi}{s^2}\left(\frac{r}{m}\right)^2, \quad (35)$$

$$f = m^2\left(4 - \frac{|s|\pi}{2} + s|s|\right) + \frac{|s| + s^2\pi/8}{4}r^2, \quad (36)$$

$$\xi = \exp\left(-\frac{\pi + 2s}{|s|}\right). \quad (37)$$

The formulas (35) and (37) explicitly illustrate the nonanalytic nature of the limit $s \to 0$ in the strong gravitational field regime for $r < 2m$.

Strategy of constructing the shell solution

Our strategy of constructing the shell solution is based on analogy to the similar problem in Einstein Theory of Gravitation. We use two Wyman solutions for different masses and glue them on the shell. Next we analyse the components of the energy-momentum tensor on the sphere which joins the solutions. Let us notice that the Minkowski space-time (with $f = 0$) can not be an internal solution because discontinuity of f produces derivatives of the delta function in $T_\mu{}^\nu$.

Infinitesimally small shell

We restrict our considerations to the most interesting case of infinitesimally small shell. In this case we can use functions (35-37) as components of the metric. Additionally, equations (25-28) posses scaling symmetry: $\gamma \to k\gamma$. This symmetry is used only inside the shell because solution (30-34) outside the shell could be identified with Schwarzschild solution only for $k = 1$. Hence the internal solution may be written in the form:

$$\gamma_- = k\left(\xi_- + \frac{\xi_-}{2|s_-|}\left(\frac{r}{\mu}\right)^2\right), \qquad \alpha_- = \frac{4\xi_-}{s_-^2}\left(\frac{r}{\mu}\right)^2, \tag{38}$$

$$f_- = \mu^2 C_- + D_- r^2, \tag{39}$$

$$\xi_- = \exp\left(-\frac{\pi + 2s_-}{|s_-|}\right), \tag{40}$$

$$C_- = 4 - \frac{|s_-|\pi}{2} + s_-|s_-|, \qquad D_- = \frac{|s_-| + s_-^2\pi/8}{4}, \tag{41}$$

where μ is a mass-like parameter.

Because we assumed that the radius R of the shell is small then outside the shell we still can use formulas (35-37) for γ_+, α_+ and f_+

$$\gamma_+ = \xi_+ + \frac{\xi_+}{2|s_+|}\left(\frac{r}{m}\right)^2, \qquad \alpha_+ = \frac{4\xi_+}{s_+^2}\left(\frac{r}{m}\right)^2, \tag{42}$$

$$f_+ = m^2 C_+ + D_+ r^2, \tag{43}$$

$$\xi_+ = \exp\left(-\frac{\pi + 2s_+}{|s_+|}\right), \tag{44}$$

$$C_+ = 4 - \frac{|s_+|\pi}{2} + s_+|s_+|, \qquad D_+ = \frac{|s_+| + s_+^2\pi/8}{4}. \tag{45}$$

Here m is the total mass of the system.

The formulas (38-45) can be combined in one set of equations that describe "the global" solution:

$$\gamma = \xi_+ + \frac{\xi_+}{2|s_+|}\left(\frac{\tilde{r}}{m}\right)^2, \tag{46}$$

where

$$\tilde{r}^2 = \begin{cases} 2m^2|s_+|\left(k\frac{\xi_-}{\xi_+} - 1\right) + k\frac{\xi_-}{\xi_+}\frac{|s_+|}{|s_-|}\frac{m^2}{\mu^2}r^2 & \text{for } r < R \\ r^2 & \text{for } r \geq R \end{cases}, \tag{47}$$

$$k = \frac{\xi_+ + \frac{\xi_+}{2|s_+|}\left(\frac{R}{m}\right)^2}{\xi_- + \frac{\xi_-}{2|s_-|}\left(\frac{R}{\mu}\right)^2}. \tag{48}$$

This choice of k ensures continuity of γ function. The radius of the shell is denoted by R. The field equations are second order therefore the functions γ and f have to be C^1. In the equations we have the first derivative of α and second derivatives of γ and f which produce δ function on the shell, in the energy-momentum tensor.

Functions α and f have the form

$$\alpha = 4\frac{\xi_-}{{s_-}^2}\frac{r^2}{\mu^2} + 4\left(\frac{\xi_+}{{s_+}^2 m^2} - \frac{\xi_-}{{s_-}^2\mu^2}\right) r^2 \Theta(r-R), \tag{49}$$

$$f = \begin{cases} f_- & \text{for } r < R \\ f_+ & \text{for } r \geq R \end{cases}, \tag{50}$$

$$\mu^2 C_- + D_- R^2 = m^2 C_+ + D_+ R^2, \tag{51}$$

where $|s_-| < 1$, $|s_+| < 1$. and Θ denotes the step function. Equation (51) ensures continuity of the function f. Finally substitution of (46-50) into field equations gives the energy-momentum tensor:

$$T_{\phi\theta} = -T_{\theta\phi} = \beta sin^2\theta T^\phi_\theta + f sin\theta T^\phi_\phi,$$

which has completely antisymmetric off diagonal components.

Energy - momentum tensor of the shell

At the beginning let us compare the physical "antisymmetric" component $\tau = T_\theta{}^\phi$ with the matter density $\rho = T_0{}^0$. From equations (25) and (28) we find the components of the energy-momentum tensor $T_\mu{}^\nu$:

$$\rho = \frac{1}{8\pi}\left(\frac{-A''}{2\alpha} + \frac{A'\alpha'}{4\alpha^2}\right), \tag{52}$$

$$\tau = \frac{\sin\vartheta}{8\pi}\left(\frac{B'}{2\alpha} - \frac{B\alpha'}{4\alpha^2}\right). \tag{53}$$

In the case of a small shell ($R \ll 2m$) we get:

$$max|\frac{\tau}{\rho}| = \left|\frac{B\alpha' - 2\alpha B'}{A'\alpha' - 2\alpha A''}\right| \rightarrow \frac{1}{|2D_+ - 2(D_+ - D_-)\omega|}, \tag{54}$$

where

$$\omega = \frac{1+\kappa}{1-\kappa}, \quad \kappa = \frac{s_+^2 m^2 \xi_-}{s_-^2 \mu^2 \xi_+}. \tag{55}$$

For $R \ll 2m$ a constant s_- approaches some non zero value determined by the equation (51). Moreover, because of the exponential vanishing of ξ_+ in the described limit we have $\kappa \to \infty$ and $\omega \to -1$. Hence

$$max|\frac{\tau}{\rho}| \to \frac{1}{|2D_-|} > \frac{2}{1+\pi/8} \simeq 1.44. \qquad (56)$$

Summary

- We constructed in the fully nonlinear theory static spherically symmetric solution in the form of shell (which is located below the General Relativity horizon $R << 2m$).

- We have shown that the energy-momentum tensor of small static and spherically symmetric shell contains large amount of the antisymmetric component. On the other hand, for all known form of matter energy-momentum tensor is purely symmetric. Inequality (56 and 58) does not predict that antisymmetric component of $T_{\mu\nu}$ is small enough as to be unobservable and therefore it seems to indicate the violation of natural physical conditions.

- The other possibility is that "finally collapsed object" is made of unknown form of matter (hidden in strong gravitational fields), which could be in principle candidate for dark matter in missing mass problem in cosmology.